Journal of Global Optimization **28:** 55–66, 2004.

Sufficient Optimality Condition for Vector Optimization Problems under D.C. Data

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(Received 20 August 2002; accepted in revised form 20 March 2003)

Abstract. In this paper, we establish sufficient optimality conditions for D.C. vector optimization problems. We also give an application to vector fractional mathematical programming in a ordred separable Hilbert space.

Mathematics Subject Classification (1991): Primary 90C29; Secondary 49K30

Key words: Convex mapping, D.C. mapping, Diff-Max mapping, Local weak minimal solution, Optimality condition, Subdifferential

1. Introduction

A lot of research has been carried out in multiobjective optimization problems [1,3,7,8,13]. Corley [3] has given optimality conditions for convex and nonvonvex multi-objective problems in terms of Clarke derivative. Luc [7] also gives optimality conditions when the data are upper semidifferentiable. Taa [13] studied optimality conditions in terms of Lagrange–Fritz–John and Lagrange–Karush-Kuhn–Tucker multipliers for nonsmooth and nonconvex vector mathematical programming with the existence of the Hadamard directional derivatives of objective and constraint functions.

In this paper, we are concerned with the vector optimization problem

$$(P):\begin{cases} Y^+ - \text{Minimize } f(x) - g(x) \\ \text{subject to } : h(x) - k(x) \in -Z^+ \end{cases}$$

where X, Y and Z are Banach spaces, f, g: $X \to Y$ and h, k: $X \to Z$ are convex, proper and lower semi-continuous mappings and $Y^+ \subset Y$ and $Z^+ \subset Z$ are pointed, convex and closed cones with nonempty interiors.

In [4], Hiriart Urruty studied a special case of (P);

$$\begin{cases} \text{Min } f(x) - g(x) \\ \text{subject to: } x \in X \end{cases}$$

where f and g are convex, proper and lower semi-continuous functions. He proved that sufficient optimality conditions can be derived either from the Diff-Max notion, which means that each point of the effective domain is a local maximum for the subdifferential according to the inclusion relation, or from the ϵ-subdifferential.

In this paper, we somewhat extend Hiriart Urruty's findings by seeing if they are valid for larger class of problems with D.C. data. To show up sufficient optimality conditions for the vector optimization problem (P), our approach consists of using extensions of both the Diff-Max notion and the ϵ-subdifferential, for convex mappings.

The outline of the paper is as follows: preliminary results are described in Section 2; the main result is given in Section 3; Sections 4 is reserved for an application to vector fractional mathematical programming in a ordered Hilbert spaces.

2. Preliminaries

Throughout this paper, X, Y, Z and W are Banach spaces whose topological dual spaces are X^*, Y^*, Z^* and W^* respectively. Let $Y^+ \subset Y$ (resp. $Z^+ \subset Z$) be a pointed ($Y^+ \cap -Y^+ = \{0\}$), convex and closed cones with nonempty interior introducing a partial order in Y (resp. in Z) defined by

$$y_1 \leqslant_Y y_2 \Leftrightarrow y_2 \in y_1 + Y^+.$$

Let S be a nonempty subset of Y; $\bar{y} \in S$ is said to be a Pareto (resp. a weak Pareto) minimal vector of S with respect to Y^+ if

$$S \subset \bar{y} + (Y \backslash -Y^+) \cup \{0\}$$

(resp. $S \subset \bar{y} + Y \backslash -Int\, Y^+$), where *Int* denotes the topological interior. The negative polar cone $(Y^+)^\circ$ of Y^+ is defined as

$$(Y^+)^\circ = \{y^* \in Y^* : \langle y^*, y \rangle \leqslant 0 \text{ for all} y \in Y^+\},$$

where $\langle .,. \rangle$ is the dual pairs.

Given a mapping $f : X \to Y$, the epigraph of φ is defined by

$$\text{epi}(f) = \{(x, y) \in X \times Y : y \in f(x) + Y^+\}.$$

Since convexity plays an important role in the following investigations, recall the concept of cone-convex mappings.

The mapping f is said to be Y^+-convex if for every $\alpha \in [0, 1]$ and $x_1, x_2 \in X$

$$\alpha f(x_1) + (1 - \alpha) f(x_2) \in f(\alpha x_1 + (1 - \alpha) x_2) + Y^+.$$

DEFINITION 2.1. A mapping h: $X \to Y$ is said to be Y^+-D.C. if there exists two Y^+-convex mappings f and g such that:

$$h(x) = f(x) - g(x) \quad \forall x \in X.$$

Let us recall the definition of lower semi-continuity of mappings introduced by Penot and Thèra [11].

DEFINITION 2.2. [11] A mapping $f: X \to Y$ is said to be lower semicontinuous (l.s.c) at $\bar{x} \in X$, if for any neighborhood V of zero and for any $b \in Y$ satisfying $b \leqslant_Y f(\bar{x})$, there exists a neighborhood U of $\bar{x}$ in X such that

$$f(U) \subset b + V + (Y^+ \cup \{+\infty\}).$$

In [16], Valadier introduced the subdifferential of Y^+-convex mappings.

DEFINITION 2.3. [16] Let $f: X \to Y \cup \{+\infty\}$ be a Y^+-convex mapping, the subdifferential of f at $\bar{x} \in domf$ is given by

$$\partial^v f(\bar{x}) = \{T \in L(X,Y) : T(h) \leqslant_Y f(\bar{x}+h) - f(\bar{x}) \quad \forall h \in X\}.$$

REMARK 2.1. 1. Let $f: X \to Y \cup \{+\infty\}$ be a Y^+-convex mapping. If f is also continuous at $\bar{x}$; then

$$\partial^v f(\bar{x}) \neq \emptyset.$$

(2.) When f is a convex function, $\partial^v f(\bar{x})$ (respectively, the lower semicontinuity) reduces to the well known subdifferential

$$\partial f(\bar{x}) = \partial_{A,C} f(\bar{x}) = \{x^* \in X^* : f(x) - f(\bar{x}) \geqslant \langle x^*, x - \bar{x} \rangle \quad \text{for all } x \in X\}.$$

(respectively. the usual lower semicontinuity).

For all the sequel, we shall need the following definition.

DEFINITION 2.4. [12] Let $f: X \to Y \cup \{+\infty\}$ be a Y^+-convex mapping. f is said to be subdifferentialy regular at $\bar{x}$ if for all $y^* \in (-Y^+)^\circ$ one has

$$y^* \circ \partial^v f(\bar{x}) = \partial(y_1^* \circ f)(\bar{x}).$$

We denote by

$$F = \{x \in X : h(x) - k(x) \in -Z^+\}, \tag{1}$$

the feasible set of (P). Consider the set

$$(f-g)(F) := \{f(x) - g(x) : x \in F\}.$$

$\bar{x} \in F$ is an efficient (resp. weak efficient) solution of (P_1) if $(f-g)(\bar{x})$ is a Pareto (resp. weak Pareto) minimal vector of $(f-g)(F)$.

$\bar{x} \in F$ is a local efficient (resp. weak local efficient) solution of (P) if there exists a neighborhood V of $\bar{x}$ such that $(f-g)(\bar{x})$ is a Pareto (resp. weak Pareto) minimal vector of $(f-g)(F \cap V)$.

The following result has been proved by Attouch and Brezis [2] in the Banach space setting and by Rodregues and Simons [13] in the case of the Frechet space.

THEOREM 2.1. (2). *Assume that* $\Psi_1, \Psi_2 : X \to \mathbb{R} \cup \{+\infty\}$ *are convex, lower semicontinuous and proper and that* $\mathbb{R}^+(dom(\Psi_1) - dom(\Psi_2))$ *is closed vector subspace of* X. *Then*

$$\partial(\Psi_1 + \Psi_2)(x) = \partial\Psi_1(x) + \partial\Psi_2(x).$$

3. Sufficient optimality conditions

In this section, we conserve the notations previously given and we give optimality conditions for (P) in terms of Lagrange–Fritz–John multipliers. There are two different approaches.

3.1. SUFFICIENT OPTIMALITY CONDITIONS VIA THE DIFF-MAX NOTION

A concept already announced in the Introduction is now formally defined.

DEFINITION 3.1. Let $f : X \to Y \cup \{+\infty\}$ be a Y^+-convex mapping and $\bar{x} \in dom f$. f is said to be Diff-Max at $\bar{x}$ if there exists a neighborhood U of $\bar{x}$ such that, for every $x \in U$; we have

$$\partial^v f(x) \subset \partial^v f(\bar{x}).$$

Note that a similar definition was given by Michelot [9] when $Y = \mathbb{R}$.

EXAMPLE 3.1. When Y is a Lattice space and $f : X \to Y \cup \{+\infty\}$ is an Y^+-convex mapping defined by

$$f(x) = \max_{i=1}^{n}(\langle l_i, x\rangle + b_i)$$

where $l_i \in L(X, Y)$ and $b_i \in Y$ for all $i \in \{1, 2, \ldots, n\}$. We have

$$\partial^v f(x) = co\{l_i : i \in I(x)\}, \text{ with} I(x) := \{i : f(x) = \langle l_i, x\rangle + b_i\}.$$

It is easy to see that f is a Diff-Max mapping.

EXAMPLE 3.2. Let $\|\cdot\|$ be an arbitrary norm of X and let $f : X \to \mathbb{R} \times \mathbb{R}$ such that

$$f(x) := (\|x\|, \|x\|) \textit{ for all} x \in X.$$

On the one hand, direct calculus yields that f is $\mathbb{R}^2_+$-convex and that

$$\partial f(0) = \mathbb{B}^*_X \times \mathbb{B}^*_X. \tag{2}$$

On the other hand, observing that f is 1-Lipschitz, one has

$$\partial f(x) \subset \mathbb{B}_X^* \times \mathbb{B}_X^*. \tag{3}$$

Then, combining (2) and (3), we obtain

$$\partial f(x) \subset \partial f(0) \text{ for all } x \in X;$$

Consequently, f is Diff-Max at 0.

We come now on to the theorem of this section.

THEOREM 3.1. *Let $\bar{x} \in F$. Assume that g and k are Diff-Max at $\bar{x}$. If in addition, there exist $y_1^* \in (-Y^+)^\circ \backslash \{0\}$ and $y_2^* \in (-Z^+)^\circ$ such that $y_2^*(h(\bar{x}) - k(\bar{x})) = 0$ and*

$$y_1^* \circ \partial^v g(\bar{x}) + y_2^* \circ \partial^v k(\bar{x}) \in \partial(y_1^* \circ f + y_2^* \circ h)(\bar{x}). \tag{4}$$

Then $\bar{x}$ is a local weak minimal solution of (P).

Proof. Let $\bar{x} \in F$. Since g and k are Diff-Max at $\bar{x}$ there exists a neighborhood U of $\bar{x}$ such that

$$\partial^v g(x) \subset \partial^v g(\bar{x}) \text{ and } \partial^v k(x) \subset \partial^v k(\bar{x})$$

for every $x \in U$.
Let $x \in U$ and consider $T^* \in \partial^v g(x)$ and $L^* \in \partial^v k(x)$. By definition,

$$g(y) \geqslant g(x) + \langle T^*, y - x\rangle \text{ and } k(y) \geqslant k(x) + \langle L^*, y - x\rangle \quad \forall y \in X.$$

If we fix $y = \bar{x}$, we get

$$g(\bar{x} \geqslant g(x) + \langle T^*, \bar{x} - x\rangle \text{ and } k(\bar{x}) \geqslant k(x) + \langle L^*, \bar{x} - x\rangle. \tag{5}$$

Moreover, by assumption there exist $y_1^* \in (-Y^+)^\circ \backslash \{0\}$ and $y_2^* \in (-Z^+)^\circ$ such that

$$y_2^* * h(\bar{x} - k(\bar{x})) = 0 \tag{6}$$

and

$$y_1^* \circ T^* + y_2^* \circ L^* \in \partial(y_1^* \circ f + y_2^* \circ h)(\bar{x}).$$

Which implies

$$y_1^* \circ f(x) + y_2^* \circ h(x) \geqslant y_1^* \circ f(\bar{x}) + y_2^* \circ h(\bar{x}) + \langle y_1^* \circ T^* + y_2^* \circ L^*, x - \bar{x}\rangle. \tag{7}$$

for all $x \in F \cap U$.
Combining (5), (6) and (7) yields

$$y_1^* \circ (f(x) - f(\bar{x})) - y_1^* \circ (g(x) - g(\bar{x})) + y_2^* \circ (h(x) - k(x)) \geqslant 0.$$

Since $h(x)-k(x)\in -Z^{+}$ and $y_2^*\in(-Z^{+})^{\circ}$, it follows that

$$y_2^*\circ(h(x)-k(x))\leqslant 0.$$

Consequently,

$$y_1^*\circ[(f(x)-f(\bar{x}))-(g(x)-g(\bar{x}))]\geqslant 0 \qquad \forall x\in U\cap F.$$

By the fact that $y_1^*\in(-Y^{+})^{\circ}\backslash\{0\}$, if follows that $\bar{x}$ is a local weak minimal solution of (P). The proof is thus complete. □

COROLLARY 3.2. *Let $\bar{x}\in F$. Suppose that g and k are Diff-Max and subdifferentialy regular at $\bar{x}$. If in addition, there exist $y_1^*\in(-Y^{+})^{\circ}\backslash\{0\}$ and $y_2^*\in(-Z^{+})^{\circ}$ such that $y_2^*(h(\bar{x})-k(\bar{x}))=0$ and*

$$\partial(y_1^*\circ g)(\bar{x}+\partial(y_2^*\circ k)(\bar{x})\in\partial(y_1^*\circ f+y_2^*\circ h)(\bar{x}).$$

Then $\bar{x}$ is a local weak minimal solution of (P).

Let A be a continuous linear operator from X into W and C be a nonempty closed convex subset of X. With few and simple computations, we can formulate the necessary optimality condition for the problem

$$(P_1):\begin{cases} Y^{+}-\text{Minimize} f(x)-g(x) \\ \qquad x\in C \\ \text{subject to:} \quad Ax=b \\ \quad h(x)-k(x)\in -Z^{+} \end{cases}$$

Using Theorem 3.1, we deduce the following result.

THEOREM 3.3. *Let $\bar{x}\in F$. Suppose that g and k are Diff-Max at $\bar{x}$ and the range of A is closed: If in addition, there exist $y_1^*\in(-Y^{+})^{\circ}\backslash\{0\}$ and $y_2^*\in(-Z^{+})^{\circ}$ such that $y_2^*(h(\bar{x})-k(\bar{x}))=0$ and*

$$y_1^*\circ\partial^v g(\bar{x})+y_2^*\circ\partial^v k(\bar{x})\in\partial(y_1^*\circ f)(\bar{x})+\partial(y_2^*\circ h)(\bar{x})+N_C(\bar{x})+rang(A^*).$$

Then $\bar{x}$ is a local weak minimal solution of (P_1).

Proof. Since the range of A is closed, by Lemma 2.4 (i) of Jeyakumar and Wolkowicz [6], we have $N_E(\bar{x})=rang(A^*)$. By the fact that

$$\partial(y_1^*\circ f)(\bar{x})+\partial(y_2^*\circ h)(\bar{x})+\partial\delta_C(\bar{x})+\partial\delta_E(\bar{x})\subset\partial(y_1^*\circ f+y_2^*\circ h+\delta_{C\subset E})(\bar{x}),$$

the proof is clear. □

Similarly, when $Y=\mathbb{R}^p$ and $Z=\mathbb{R}^m$, we deduce the following results.

COROLLARY 3.4. *Let $\bar{x} \in F$. Suppose that g_i and k_j are Diff-Max at $\bar{x}$ for $i=1,\ldots,p$ and $j=1,\ldots,m$ and the range of A is closed. If in addition, there exist $\alpha=(\alpha_1,\ldots,\alpha_p) \in \mathbb{R}_+^p \backslash \{0,\ldots,0\}$ and $\beta=(\beta_1,\ldots\beta_m) \in \mathbb{R}_+^m$ such that*

$$\beta_j(h_j(\bar{x})=k_j(\bar{x}))=0\, j=1,\ldots,p$$

and

$$\sum_{i=1}^{p}\alpha_i\partial g_i(\bar{x})+\sum_{j=1}^{m}\beta_j\partial k_j(\bar{x}) \in \sum_{i=1}^{p}\alpha_i\partial f_i(\bar{x})+\sum_{j=1}^{m}\beta_j\partial h_j(\bar{x})+N_C(\bar{x})+rang(A^*).$$

Then $\bar{x}$ is a local weak minimal solution of (P_1).

Let us recall the following concept introduced by Hiriart-Urruty [4]. Given a function ψ: $X \rightarrow \mathbb{R}$. ψ is polyhedral (or piecewise affine) convex function if

$$\psi(x)=\max\{\langle a_i^*, x\rangle + d_i : i=1,\ldots,q\}$$

for all $x \in X$, where $a_1^*,\ldots,a_q^*$ are in X^* and $d_1,\ldots,d_q$ are real numbers.

COROLLARY 3.5. (15) *Let $\bar{x} \in F$. Suppose that g_i and k_j are polyhedrals, $i=1,\ldots,p$ and $j=1,\ldots,m$ and the range of A is closed: If in addition, there exist $\alpha=(\alpha_1,\ldots,\alpha_p) \in \mathbb{R}_+^P \backslash \{0,\ldots,0\}$ and $\beta=(\beta_1,\ldots,\beta_m) \in \mathbb{R}_+^m$ such that*

$$\beta_j(h_j(\bar{x})-k_j(\bar{x}))=0\, j=1,\ldots,p$$

and

$$\sum_{i=1}^{p}\alpha_i\partial g_i(\bar{x})+\sum_{j=1}^{m}\beta_j\partial k_j(\bar{x}) \in \sum_{i=1}^{p}\alpha_i\partial f_i(\bar{x})+\sum_{j=1}^{m}\beta_j\partial h_j(\bar{x})+N_C(\bar{x})+rang(A^*).$$

Then $\bar{x}$ is a local weak minimal solution of (P_1).

Proof. From Hiriart-Urruty [4], since g_i and k_j are polyhedral functions, there exists a neighborhood U of $\bar{x}$ such that

$$\partial g_i(x) \subset \partial g_i(\bar{x}) \text{ and } \partial k_j(x) \subset k_j(\bar{x})$$

for all $x \in U$. Thus, g_i and h_j are Diff-Max functions at $\bar{x}$ for all i,j. Using Corollary 3.4, the proof is completed. □

3.2. SUFFICIENT OPTIMALITY CONDITIONS VIA THE VECTOR ϵ-SUBDIFERENTIAL

Let $\epsilon \in int(Y^+)$. By analogy to the scalar case, the vector ϵ-subdifferential of f at $\bar{x} \in X$ is defined by

$$\partial^v_\epsilon f(\bar{x}) = \{T \in L(X,Y) : T(h) - \epsilon \leqslant_Y f(\bar{x}+h) - f(\bar{x}) \quad \forall h \in X\}.$$

The particular case $\epsilon = 0$, corresponds to $\partial^v f(\bar{x})$.
When $Y = \mathbb{R}$ and $\epsilon \in \mathbb{R}^*_+$, $\partial^v_\epsilon f(\bar{x})$ reduces to the well known ϵ-subdifferential

$$\partial_\epsilon f(\bar{x}) = \{x^* \in X^* : f(x) - f(\bar{x}) \geqslant \langle x^*, x - \bar{x}\rangle - \epsilon \text{ for all } x \in X\}.$$

We will need the following result due to J. B. Hiriart-Urruty, M. Moussaoui, A. Seeger and M. Volle [6].

THEOREM 3.6. *Suppose that* $\Psi_1, \Psi_2 : X \to \mathbb{R} \cup \{+\infty\}$ *are convex, proper and lower semicontinuous and* $\bar{x} \in \mathrm{dom}(\Psi_1) \cap \mathrm{dom}(\Psi_2)$. *Then, for all* $\varepsilon > 0$, *one has*

$$\partial_\varepsilon(\Psi_1 + \Psi_2)(\bar{x}) = cl\left(\bigcap_{\substack{\alpha \geqslant 0, \beta \geqslant 0 \\ \alpha+\beta=\varepsilon}} \partial_\alpha \Psi_1(\bar{x}) + \partial_\beta \Psi_2(\bar{x})\right),$$

where "cl" stands for topological closure operation with respect to weak star topology $\sigma(X^*, X)$.

THEOREM 3.7. *Let* $\bar{x} \in F$. *Assume that there exist* $y_1^* \in (-Y^+)^\circ \backslash \{0\}$ *and* $y_2^* \in (-Z^+)^\circ$ *such that*

$$y_2^*(h(\bar{x}) - k(\bar{x})) = 0 \tag{8}$$

and for all $\alpha, \beta \in \mathbb{R}^*_+$,

$$cl\left(\bigcap_{\alpha \geqslant 0, \beta \geqslant 0} \partial_\alpha(y_1^* \circ g)(\bar{x}) + \partial_\beta(y_2^* \circ k)(\bar{x})\right) \subset \partial_{\alpha+\beta}(y_1^* \circ f + y_2^* \circ h)(\bar{x}).$$

Then $\bar{x}$ *is a local weak minimal solution of* (P).

Proof. Using Lemma 3.6, we get

$$cl\left(\bigcap_{\alpha \geqslant 0, \beta \geqslant 0} \partial_\alpha(y_1^* \circ g)(\bar{x}) + \partial_\beta(y_2^* \circ k)(\bar{x})\right) = \partial_{\alpha+\beta}(y_1^* \circ g + y_2^* \circ k)(\bar{x}).$$

Let $t^* \in \partial_{\alpha+\beta}(y_1^* \circ g + y_2^* \circ k)(\bar{x})$. By definition, for all and $x \in X$

$$-\langle t^*, \bar{x} - x\rangle + \alpha + \beta \geqslant (y_1^* \circ f + y_2^* \circ h)(\bar{x}) - (y_1^* \circ g + y_2^* \circ k)(x). \tag{9}$$

By assumption, for all $\alpha,\beta\in\mathbb{R}_+^*$ and $x\in X$

$$-\langle t^*,\bar{x}-x\rangle+\alpha+\beta\geqslant(y_1^*\circ f+y_2^*\circ h)(\bar{x})-(y_1^*\circ f+y_2^*\circ h)(x). \tag{10}$$

Since (9) implies (10) for all $\alpha,\beta\in\mathbb{R}_+^*$ and $x\in X$, one gets

$$y_1^*\circ(f(x)-f(\bar{x}))-y_1^*\circ(g(x)-g(\bar{x}))+y_2^*\circ(h(x)-k(x))-y_2^*(h(\bar{x})-k(\bar{x}))\geqslant 0. \tag{11}$$

Since $h(x)-k(x)\in -Z^+$ and $y_2^*\in(-Z^+)^\circ$, one has

$$y_2^*\circ(h(x)-k(x))\leqslant 0. \tag{12}$$

Combining (8), (11) and (12) we obtain

$$y_1^*\circ[(f(x)-f(\bar{x}))-(g(x)-g(\bar{x}))]\geqslant 0 \quad \forall x\in F.$$

By the fact that $y_1^*\in(-Y^+)^\circ\backslash\{0\}$, if follows that $\bar{x}$ is a weak minimal solution of (P).

The proof is thus complete. □

Let $\lambda\in int(Y^+)$ and $\mu\in int(Z^+)$.

COROLLARY 3.8. *Let $\bar{x}\in F$. Suppose that g and k are subdifferentialy regular at $\bar{x}$. If in addition, there exist $y_1^*\in(-Y^+)^\circ\backslash\{0\}$ and $y_2^*\in(-Z^+)^\circ$ such that $\mathbb{R}^+(dom(y_1^*\circ g)-dom(y_2^*\circ k))$ is closed vector subspace of X, $y_2^*(h(\bar{x})-k(\bar{x}))=0$ and*

$$cl\left(\bigcap_{\alpha=y_1^*\circ\lambda+y_2^*\circ\mu} y_1^*\circ\partial_\lambda^v g(\bar{x})+y_2^*\circ\partial_\mu^v k(\bar{x})\right)\subset\partial_\alpha(y_1^*\circ f+y_2^*\circ h)(\bar{x}).$$

Then $\bar{x}$ is a local weak minimal solution of (P).

Proof. Since g and k are subdifferentialy regular at $\bar{x}$, we have that

$$y_1^*\circ\partial_\lambda^v g(\bar{x})=\partial_{\delta 1}(y_1^*\circ g)(\bar{x}) \text{ and } y_2^*\circ\partial_\mu^v k(\bar{x})=\partial_{\delta 2}(y_2^*\circ k)(\bar{x})$$

where $\delta_1=y_1^*\circ\lambda$ and $\delta_2=y_2^*\circ\mu$. Consequently,

$$y_1^*\circ\partial_\lambda^v g(\bar{x})+y_2^*\circ\partial_\mu^v(\bar{x})=\partial_{\delta 1}(y_1^*\circ g)(\bar{x})+\partial_{\delta 2}(y_2^*\circ k)(\bar{x}). \tag{13}$$

Combining (13) and Theorem 3.7, the proof is finished. □

4. Application

In this section, we give an application to vector fractional mathematical programming. Consider H as a separable Hilbert space ordered by a closed convex cone

$$H^+ = \{x \in X / \langle e_i, x\rangle \geqslant 0 \text{ for all } i\}$$

with $(e_i)_i$ being an orthogonal base. Let $f: X \to H^+, g: X \to H^+$; be given H^+-convex and lower semicontinuous mappings such that $g_i(x) = \langle e_i, g(x)\rangle \neq 0$. We denote by ϕ the mapping defined as follows

$$\phi(x) := \frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_i(x)}{g_i(x)}, \ldots\right).$$

We suppose that there exists $x_0 \in X$ such that $\phi(x_0) \in H$. Under these assumptions, we investigate the vector optimization problem

$$(P_F): \begin{cases} H^+ - Minimize \phi(x) \\ \quad x \in C, \\ \text{subject to:} \quad Ax = b \\ \quad h(x) - k(x) \in -Z^+, \end{cases}$$

where C, A, h and k are as in problem (P_1).

We will need the following lemma.

LEMMA 4.1. *Let $\bar{x}$ be a feasible point of problem (P_F). $\bar{x}$ is a local weak minimal solution of (P_F) if and only if $\bar{x}$ is a local weak minimal solution of the following problem*

$$\begin{cases} H^+ - minimize(f_1(x) - \phi_1(\bar{x})g_1(x), \ldots, f_i(x) - \phi_i(\bar{x})g_i(x), \ldots) \\ \quad x \in C, \\ subject\ to: \quad Ax = b, \\ \quad h(x) - k(x) \in -Z^+ \end{cases}$$

where $\phi_i(\bar{x}) = \frac{f_i(\bar{x})}{g_i(\bar{x})}$.

Proof. Let $\bar{x}$ be a local weak minimal solution of (P_F). It is easy to see that $\phi(\bar{x}) \in H$. If there exists $x_1 \in \bar{x} + \mathbb{B}_X$ such that $x_1 \in C, Ax_1 = b, h(x_1) - k(x_1) \in -Z^+$ and

$$(f_i(x_1) - \phi_i(\bar{x})g_i(x_1)) - (f_i(\bar{x}) - \phi_i(\bar{x})g_i(\bar{x})) \in -Int(H^+).$$

Since $f_i(\bar{x}) - \phi_i(\bar{x})g_i(\bar{x}) = 0$, one has

$$\frac{f_i(x_1)}{g_i(x_1)} - \frac{f_i(\bar{x})}{g_i(\bar{x})} \in -Int(H^+),$$

which contradicts the fact that $\bar{x}$ is a local weak minimal solution of (P_F).

The converse implication can be proved in the similar way. The proof is thus completed. □

Using Theorem 3.1 and Lemma 4.1, we have the following result.

THEOREM 4.2. *Let $\bar{x} \in F$ such that $\phi(\bar{x}) \in H$. Assume that g and k are Diff-Max at $\bar{x}$. If in addition, there exist $y^* \in H^+ \setminus \{0\}$ and $z^* \in (-Z^+)^\circ$ such that*

$$z^*(h(\bar{x}) - k(\bar{x})) = 0$$

and

$$\sum_{i=1}^{\infty} \phi_i(\bar{x}) y_i^* \partial g_i(\bar{x}) + z^* \circ \partial k(\bar{u}) \in \partial(y^* \circ f + z^* \circ h + \delta_{C \cap E})(\bar{x}).$$

Then $\bar{x}$ x is a local weak minimal solution of (P_F).

5. Acknowledgment

Thanks are due to the anonymous referees for the careful reading and the improvements they bring to our paper. We would like to thank also Pr. T. Amahroq, Pr. A. Radi and Pr. A. Taa for remarks which improved the original version of this work.

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